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Components in Graphs of Diagram Groups over the Union of Two Semigroup Presentations of Integers

(Kumpulan-Kumpulan Gambar Rajah Atas Kesatuan Dua Persembahan Semikumpulan Bagi Integer)

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ABSTRACT

Given any semigroup presentation, we may obtain the diagram group. The purpose of this paper is to determine the graphs $\Gamma_n(P)$, $(n \in N)$, which are obtained from diagram group for the union of two semigroup presentations of integers with s and t different initial generators. The number of vertices and edges in these graphs will be computed.

Keywords: Diagram groups; generators; initial generators; relation; semigroup presentation

ABSTRAK

Diberi sebarang persembahan semikumpulan, kita boleh peroleh kumpulan gambar rajah. Tujuan kertas ini ialah untuk menentukan graf-graf $\Gamma_n(P)$, $(n \in N)$ yang diperoleh daripada kumpulan gambar rajah untuk kesatuan dua persembahan semikumpulan dengan s dan t penjana awal yang berbeza. Bilangan bucu dan tepi dalam graf-graf ini akan dihitung.

Kata kunci: Hubungan; kumpulan gambar rajah; penjana; penjana awal; persembahan semikumpulan

INTRODUCTION

In our previous work, we obtained the general formula of the component in graphs for semigroup presentation $P = \langle x, y, z | x = y, y = z, x = z\rangle$ and also we obtained the lifts of spanning trees of semigroup presentation $P = \langle x, y, z| x = y, y = z, x = z\rangle$ (Gheisari & Ahmad 2009, 2010). In this research, we determined some properties of component in graphs associated with the semigroup presentations of the union of two semigroup presentations of integers with s and t different initial generators by adding a relation.

Let $P_1 = \langle x_1, x_2, \ldots, x_s | x_i = x_j, 1 \le i < j \le s\rangle$, and $P_2 = \langle a_1, a_2, \ldots, a_t | a_i = a_j, 1 \le i < j \le t\rangle$ be the semigroup presentations. Now we consider the new semigroup presentation $P = \langle x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t | x_i = x_j, 1 \le i < j \le s, a_i = a_j, 1\le i < j \le t\rangle$ obtained from union of initial generators and relations of P_1 and P_2 by adding a relation $x_1 = a_1$. (Guba & Sapir (1997); Kilibarda (1994,1997); Pride (1995)).

In the materials and method section, we will determine the graphs $\Gamma_n(P)$, $(n \in N)$ where $N = \{1,2,3,\ldots\}$ obtained from the semigroup presentation $P = \langle x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t | x_i = x_j, 1 \le i < j \le s, a_i = a_j, 1 \le i < j \le t\rangle$.

In the result and discussion section, we computed the total number of vertices and edges in the graphs $\Gamma_n(P)$.

MATERIALS AND METHODS

Let $P = \langle x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t | x_i = x_j, 1 \le i < j \le s, a_i = a_j, 1 \le i < j \le t\rangle$ be a semigroup presentation. Associated with any semigroup presentation $S = \langle X|R\rangle$ we have a graph Γ where the vertices are word on X and the edges are the form $e = (T_1, T_\varepsilon \to R_{-\varepsilon}, T_2)$ such that $\iota(e) = T_1 R_\varepsilon T_2$, $\tau(e) = (T_1 R_{-\varepsilon} T_2)$. The graph obtained from S is collections of subgraphs Γ_n. Note that the graph $\Gamma(P_1)$ obtained from P_1 is just a collection of subgraphs $\Gamma_n(P_1)$ where $\Gamma_n(P_1)$ contains all vertices of length n and respective edges. Similarly we obtain $\Gamma_n(P_2)$ for P_2. Now for P, the graph $\Gamma_n(P) = \Gamma_n(P_1) \cup \Gamma_n(P_2) \cup \{(u, x_1 \to a_1, v)\}$ such that length $uv = n - 1$. If T_n is a vertex in $\Gamma_n(P)$, then $T_n g$, where $(g \in \{x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t\})$ is a vertex in $\Gamma_{k+1}(P)$. Similarly, if $(u, R_\varepsilon \to R_{-\varepsilon}, v)$ is a edge in $\Gamma_n(P)$, then $(u, R_\varepsilon \to R_{-\varepsilon}, vg)$ is the respective edges in $\Gamma_{n+1}(P)$. Thus $\Gamma_{n+1}(P)$ is just $(s + t)$ copies of $\Gamma_n(P)$ together with $(s + t)$ vertices $(u, x_1 \to a_1, vg)(g \in \{x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t\})$.

For example the graph $\Gamma_1(P)(V_1, E_1)$, where $V_1 = X = \{x_1, x_2, \ldots, x_s, a_1, a_2, \ldots a_t\}$ is set of vertices in the graphs of $\Gamma_1(P)$ and let $e_{1x} = \{(1, x_i \to x_j, 1), (1 \le i < j \le s)\}$, and $e_{1a} = \{(1, a_i \to a_j, 1), (1 \le i < j \le t\}$. $E_1 = \{e_{1x} \cup e_{1a} \cup x_1 = a_1\}$ is set of edges in the graph $\Gamma_1(P)$ (Figure 1).

And $\Gamma_2(P)(V_2, E_2)$ is $(s + t)$ copies of $\Gamma_1(P)(V_1, E_1)$. Similarly we may obtain the graph for $\Gamma_n(P)$ (V_n, E_n), $(n \in N)$.

Note that $\Gamma_2(P)$ is $(s + t)$ copies of $\Gamma_1(P)$ and each vertex in each copy are joined together, respectively by considering the relation $x_1 = a_1$. Similarly, with $(s + t)$ copies of $\Gamma_2(P)$, we may obtain $\Gamma_3(P)$. Repeating similar procedures for $\Gamma_4(P)$ and so on to obtain $\Gamma_n(P)$.

Results and Discussion

Lemma Let $P = \langle x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t | x_i = x_j, 1 \le i < j \le s, a_i = a_j, 1 \le i < j \le t\rangle$ be the presentation, and u and v are two positive words on $\{x_1, x_2, \ldots x_s, a_1, a_2, \ldots a_t\}$, if length (u) = length (v) then $\pi_1(K(S), u) = \pi_1(K(S), v)$.

Proof: The proof of this lemma is similar to that of lemma 2.3 in Gheisari and Ahmad (2009), and Ahmad and Al-Odhari (2004).

Lemma Let the following semigroup presentation of integers $P_1 = \langle x_1, x_2, \ldots, x_s | x_i = x_j, 1 \le i < j \le s\rangle$. The number of vertices in $\Gamma_n(P_1)$ is $v_n = s^n$, where v_i is the number of vertices in $\Gamma_i(P_1)(i = 1,2,3,\ldots)$.

Proof: By induction on n.

Lemma Consider the semigroup presentation of integers $P_2 = \langle a_1, a_2, \ldots, a_t | a_i = a_j, 1 \le i < j \le t\rangle$. The number of vertices in $\Gamma_n(P_2)$, is $v_n = t^n$.

Proof: By induction on n.

Theorem Let the following semigroup presentation $P = \langle x_1, x_2, \ldots, x_s, a_1, a_2, \ldots, a_t | x_i = x_j, 1 \le i < j \le s, a_i = a_j, 1 \le i < j \le t\rangle$. The number of vertices in $\Gamma_n(P)$ is $v_n = (s + t)^n$, where v_i is the number of vertices in $\Gamma_i(P)(i = 1,2,3,\ldots)$.

Proof: By induction, for $k = 1$ the number of all vertices in $\Gamma_1(P)$ is $(s + t)$. Thus for $k = 1$ is true (Figure 1). Now assume $v_k = (s + t)^k$ be the number of all vertices in $\Gamma_k(P)$. We will prove that the number of all vertices in $\Gamma_{k+1}(P)$ is $v_{k+1} = (s + t)^{k+1}$. By definition $\Gamma_{k+1}(P)$ is $(s + t)$ copies of $\Gamma_k(P)$ and using the assumption, then the vertices of $\Gamma_{k+1}(P)$ is $v_{k+1} = (s + t). (s + t)^k = (s + t)^{k+1}$

Theorem The total number of edges in the graph $\Gamma_n(P)$ is

$$e_n = \begin{cases} 4e_{n-1} + 3\left(4^{n-1}\right) & \text{if}\left(s = t = 2\right) \\ \left(s+t\right)e_{n-1} + \left(s+t\right)^n + \left(s+t\right)^{n-1} & \text{if}\left(s \ne 2, t \ne 2\right) \\ \left(s+t\right)e_{n-1} + \left(s+t\right)^n & \text{if}\left(s,t \ne 2\right)\text{simultanuesly} \end{cases}$$

where e_i is the number of edges in $\Gamma_i(P)(i = 1,2,3,\ldots)$.

Proof: Case 1: If $s = t = 2$, then we have the semigroup presentations ${}^2P_1 = \langle x_1, x_2 | x_1 = x_2\rangle$, ${}^2P_2 = \langle a_1, a_2 | a_1 = a_2\rangle$. Now we consider the new semigroup presentation $P = \langle x_1, x_2, a_1, a_2 | x_1 = x_2, a_1 = a_2, x_1 = a_1\rangle$ obtained from the union of initial generators and relations of ${}^2P_1$ and ${}^2P_2$ by adding a relation $x_1 = a_1$.

Now consider the graphs of $\Gamma_1(P)$ in Figure 2, and $\Gamma_2(P)$ in Figure 3.

By definition, $\Gamma_n(P)$ is four copies of $\Gamma_{n-1}(P)$, and by considering the Figures 2 and 3, if there is e_{n-1} edges in $\Gamma_{n-1}(P)$, where e_{n-1} is the number of all edges in $\Gamma_{n-1}(P)$ then the number of edges in $\Gamma_n(P)$ is $4e_{n-1}$ plus all edges between the vertices in $\Gamma_n(P)$, which is $3\times4^{n-1}$. Thus the number of all edges in $\Gamma_n(P)$ is $e_n = 4e_{n-1} + 3\times4^{n-1}$.

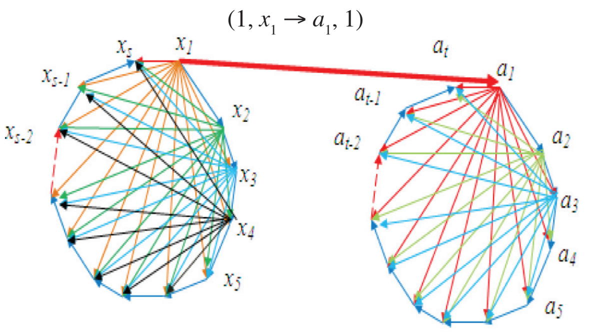

FIGURE 1. Graph of $\Gamma_1(P)$

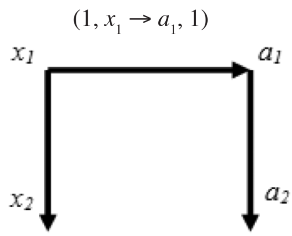

FIGURE 2. Graph of $\Gamma_1(P)$

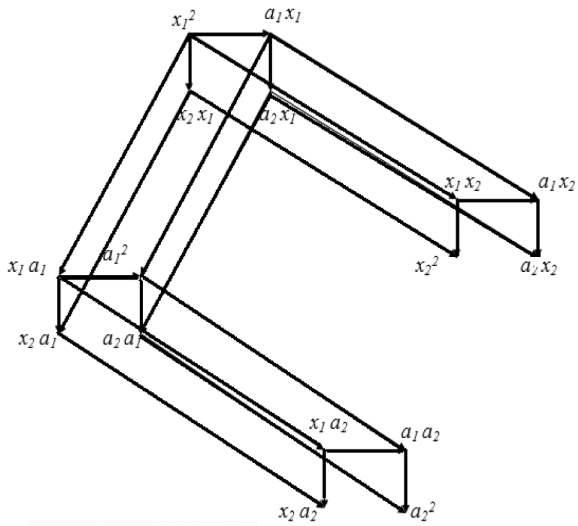

FIGURE 3. Graph of $\Gamma_2(P)$

Case 2: By definition $\Gamma_n(P)$ is $(s + t)$ copies of $\Gamma_{n-1}(P)$. Thus if there is e_{n-1} edges in $\Gamma_{n-1}(P)$, then the number of edges in $\Gamma_n(P)$ is $(s + t)\ e_{n-1}$ plus all edges between the vertices in $\Gamma_n(P)$ with considering the relation $x_1 = a_1$, which is $(s + t)^n + (s + t)^{n-1}$. Thus the number of all edges in $\Gamma_n(P)$ is $e_n = (s + t)e_{n-1} + (s + t)^n + (s + t)^{n-1}$.

Case 3: For Case 3, first if we prove that $s = 2$, $t = 3$, then for case 3 is similarly to this proof. Let the following semigroup presentations of integers ${}^2P_1 = \langle x, y, z | x = y, x = z, y = z\rangle$, and ${}^2P_2 = \langle a,b | a = b\rangle$. Now we consider the new semigroup presentation $P = \langle x,y,z,a,b | x = y, x = z, y = z, a = b, x = a\rangle$ obtained from the union of initial generators and relations of ${}^2P_1$ and ${}^2P_2$ by adding a relation $x = a$. Now consider the graphs of $\Gamma_1(P)$ in Figure 4, and $\Gamma_2(P)$ in Figure 5.

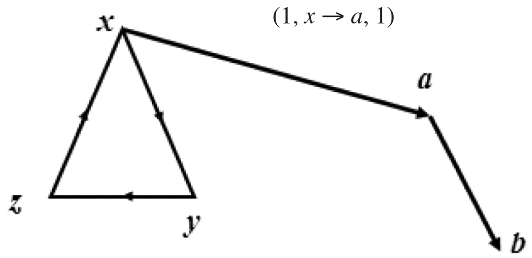

FIGURE 4. Graph of $\Gamma_1(P)$

The graph of $\Gamma_2(P)$ is just five copies of $\Gamma_1(P)$ (Figure 5).

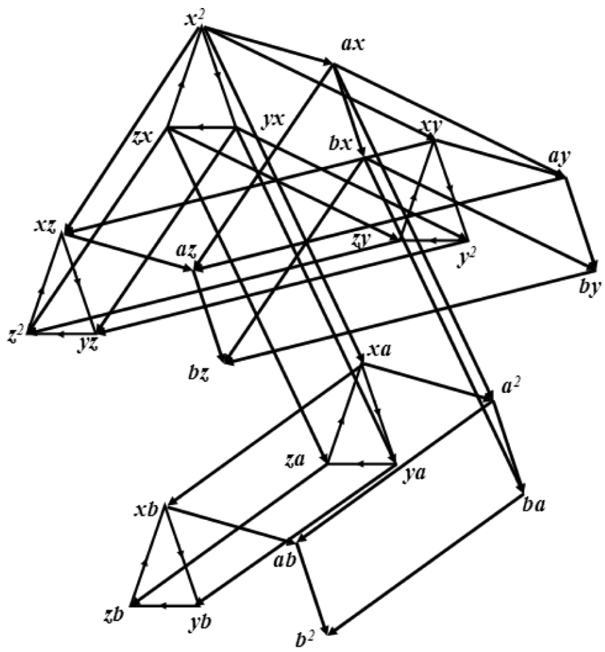

FIGURE 5. Graph of $\Gamma_2(P)$

This completes the proof. For this case we will prove that the recurrence formula of the number of all edges in $\Gamma_n(P)$ is $e_n = 5e_{n-1} + 5^n$, where e_i is the total number of edges in $\Gamma_i(P)(i = 1,2,3,\ldots)$.

By definition $\Gamma_n(P)$ is five copies of $\Gamma_{n-1}(P)$, and considering the graphs of $\Gamma_1(P)$ and $\Gamma_2(P)$ (refer to Figures 4, and 5). Thus if there is e_{n-1} edges in $\Gamma_{n-1}(P)$, then the number of edges in $\Gamma_n(P)$ is $5e_{n-1}$ plus all edges between the vertices in $\Gamma_n(P)$, which is 5^n. Thus the number of all edges in $\Gamma_n(P)$ is $e_n = 5e_{n-1} + 5^n$.

CONCLUSION

In this paper we determined the graphs $\Gamma_n(P)$, $(n \in N)$, which is obtained from union of two semigroup presentation of integers with finite different initial generators. Also we computed the number of vertices and edges of these graphs.

REFERENCES

Ahmad, A.G.B. & Al-Odhari, A.M. 2004. The graph of diagram groups constructed from natural numbers semigroup with a repeating generator. *Jour. of Inst. of Math & Com. Sci. (Math. Ser.)* 17: 67-69.

Gheisari, Y. & Ahmad, A.G.B. 2009. Spanning trees for diagram groups of semigroup presentation $P = \langle x, y, z | x = y, y = z, z = z\rangle$ using lifting method. *Proceeding of the 5th Asian Mathematical Conference, Kuala Lumpur* pp. 366-372.

Gheisari, Y. & Ahmad, A.G.B. 2010. General polynomial for component of graphs from diagram group of semigroup presentation $P = \langle x, y, z | x = y, y = z, z = z\rangle$. *Journal of International Mathematical Forum* 18: 875-880.

Guba, V. & Sapir, M. 1997. Diagram Groups. *Memoirs of the American Mathematical Society.*

Kilibarda, V. 1997. On the algebra of semigroup diagrams, *Int. Journal of Alg. and Comput.* 7: 313-338.

Kilibarda, V. 1994. On the algebra of semigroup diagram. PhD Thesis. University of Nebraska.

Pride, S.J. 1995. Geometric method in combinatorial group theory. In *Proc. of Int. Conf. on Groups,* editor by J. Fountain. *Semigroups and Formal Languages.* NY: Kluwer Publ. pp. 215-23.

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